On a Property of the Class of all Real Algebraic Numbers.

by Georg Cantor

Crelle's Journal for Mathematics, Vol. 77, pp. 258–262 (1874).

Translated by Christopher P. Grant

By a real algebraic number is generally understood a real numerical quantity ω that satisfies a nontrivial equation of the form:

$$a_0\omega^n + a_1\omega^{n-1} + \cdots + a_n = 0, \tag{1}$$

where n, $a_0, a_1, \ldots a_n$ are integers; we can take the numbers n and a_0 to be positive, the coefficients $a_0, a_1, \ldots a_n$ to have no common factor, and the equation (1) to be irreducible; with these restrictions, the equation (1) that a real algebraic number satisfies will be completely determined, according to the well-known laws of arithmetic and algebra; conversely, it is well-known that to an equation of the form (1) there belong at most as many real algebraic numbers ω that solve it as its degree n indicates. In their totality, the real algebraic numbers form a class of numerical quantities that will be denoted (ω); it itself has the property, which follows from simple observations, that in each neighborhood of an arbitrarily chosen number α there are infinitely many numbers from (ω); at first glance this might make all the more striking the observation that the class (ω) can be put in clear correspondence with the class of all positive integers ν (which we denote with the symbol (ν)), so that to each algebraic number ω there corresponds a distinct positive integer ν and, conversely, to each positive integer ν there corresponds a distinct real algebraic number ω, that, in other words, the class (ω) can be thought of in the form of an infinite rule-based sequence

$$\omega_1, \omega_2, \ldots \omega_\nu, \ldots, \tag{2}$$

in which all individuals of (ω) occur and each one of them is to be found in a certain position in (2), given by the corresponding index. Once a rule has been found according to which such an assignment can be performed, it can be modified arbitrarily; it will therefore suffice if I share in §1 that method of assignment which, it appears to me, demands the least background.

In order to give an application of this property of the class of all real algebraic numbers, I supplement §1 with §2, in which I show that, given any sequence of real numerical quantities of the form (2), one can determine numbers η in any given interval $(\alpha \ldots \beta)$ that are *not* contained in (2); if one combines the contents of both of these sections, a new proof is given of the theorem first proved by Liouville, that in any given interval $(\alpha \ldots \beta)$ there are infinitely many *transcendentals*, *i.e.* non-algebraic real numbers. Furthermore, the theorem in §2 represents the reason why classes of real numerical quantities that form a so-called continuum (for instance all the real numbers that are ≥ 0 and ≤ 1), can't be correlated with the class (ν); thus, I found the clear difference between a so-called continuum and a class of the type of the totality of all real algebraic numbers.

§1.

If we go back to equation (1), which an algebraic number ω satisfies and which, according to our restrictions, is completely determined, we can call the sum of the absolute values of the coefficients and the number $n-1$ (where n is the degree of ω) the *height* of the number ω and denote it with N; using now-common notation, we therefore have

$$N = n - 1 + |a_0| + |a_1| + \cdots + |a_n|. \tag{3}$$

According to this, the height N is for each real algebraic number a specified positive integer; conversely, for each positive integer value of N there are only a finite number of algebraic real numbers with height N; let the number of these be $\varphi(N)$; for example, $\varphi(1) = 1$; $\varphi(2) = 2$; $\varphi(3) = 4$. The numbers in the class (ω) (*i.e.*, all algebraic real numbers) can then be ordered in the following way: take as the first number ω_1 the one number with height $N = 1$; after it, let the $\varphi(2) = 2$ algebraic real numbers with height $N = 2$ follow in ascending order, and denote them ω_2, ω_3; after these, let the $\varphi(3) = 4$ numbers with height $N = 3$ follow in ascending order; in general, once all numbers from (ω) up to a certain height $N = N_1$ have been enumerated and given a specific place in this manner, let the real algebraic numbers with height $N = N_1 + 1$ follow, again in ascending order; thus, one obtains the class (ω) of all real algebraic numbers in the form:

$$\omega_1, \omega_2, \ldots \omega_\nu, \ldots$$

and with reference to this ordering can speak of the νth real algebraic number, without omitting a single member of the class (ω).

§2.

If an infinite sequence of distinct real numerical quantities

$$\omega_1, \omega_2, \ldots \omega_\nu, \ldots \tag{4}$$

(obtained according to whatever rule) is given, then in each prespecified interval $(\alpha \ldots \beta)$ a number η (and consequently infinitely many such numbers) can be specified, which does not occur in the sequence (4); this will now be proven.

To this end, we start with the interval $(\alpha \ldots \beta)$, which is arbitrarily given to us, with $\alpha < \beta$: the first two numbers in our sequence (4) that lie in the interior of this interval (with the endpoints being excluded) may be denoted α', β', with $\alpha' < \beta'$; likewise, the first two numbers in our sequence that lie in the interior of $(\alpha' \ldots \beta')$ may be denoted α'', β'', with $\alpha'' < \beta''$, and by the same method we can form a subsequent interval $(\alpha''' \ldots \beta''')$, etc. Therefore, according to the definition, $\alpha', \alpha'' \ldots$ are here specific numbers in our sequence (4) whose indices are in increasing order, and the same holds for the numbers $\beta', \beta'' \ldots$; furthermore, the numbers $\alpha', \alpha'', \ldots$ are getting bigger and bigger, and the numbers $\beta', \beta'', \ldots$ are getting smaller and smaller; the intervals $(\alpha \ldots \beta), (\alpha' \ldots \beta'), (\alpha'' \ldots \beta''), \ldots$ each include all that follow. —Here are now two cases possible.

Either the number of intervals constructed in this way is finite; let the last of them be $(\alpha^{(\nu)} \dots \beta^{(\nu)})$; since in the interior of this interval at most one number of the sequence (4) can lie, there can be assumed to be a number η in this interval that is not contained in (4), and with that the theorem is proven for this case. —

or the number of the intervals constructed in this way is infinitely large; then the numbers $\alpha, \alpha', \alpha'', \dots$, because they are continuously increasing in size without growing to infinity, have a specific limit value α^{∞}; the same holds for the numbers $\beta, \beta', \beta'', \dots$, since they are continuously decreasing in size, so let their limit value be β^{∞}; if $\alpha^{\infty} = \beta^{\infty}$ (a case that always arises with the class (ω) of all real algebraic numbers), then one can easily convince oneself, simply by looking back at the definition of the intervals, that the number $\eta = \alpha^{\infty} = \beta^{\infty}$ *cannot* be contained in our sequence[1]; however, if $\alpha^{\infty} < \beta^{\infty}$, then every number η in the interior of the interval $(\alpha^{(\infty)} \dots \beta^{(\infty)})$ or even on its boundary satisfies the requirement not to be contained in the sequence (4). —

The theorems proved in this paper admit extensions in various directions, only one of which may be mentioned here:

"If $\omega_1, \omega_2, \dots \omega_n, \dots$ is a finite or infinite sequence of linearly independent numbers (so that no equation of the form $a_1\omega_1 + a_2\omega_2 + \cdots + a_n\omega_n = 0$ with integer coefficients that don't all vanish is possible) and if one considers the class (Ω) of all those numbers Ω that can be represented as rational functions with integer coefficients of the given numbers ω, then there is in each interval $(\alpha \dots \beta)$ infinitely many numbers that are not contained in (Ω)."

In fact, one convinces oneself through reasoning similar to that in §1 that the class (ω) can be conceived in the sequence form

$$\Omega_1, \Omega_2, \dots \Omega_\nu, \dots,$$

from which follows the correctness of this theorem, in consideration of §2.

A very special case of the theorem stated here (in which the sequence $\omega_1, \omega_2, \dots \omega_n \dots$ is a finite one and the degree of the rational functions that generate the class (Ω) is provided) was proved by Mr. B. Minnigerode by reduction to Galois principles. (See Math. Annalen, Vol, 4, p. 497.)

[1]If the number η were contained in our sequence, we'd have $\eta = \omega_p$, where p is a specific index; this, however, is impossible, because ω_p does *not* lie in the interior of the interval $(\alpha^{(p)} \dots \beta^{(p)})$, while the number η lies in the interior of this interval according to its definition.